4.
Introduction to Prescriptive Analytics

Why is Decision Making difficult?

- The biggest sources of difficulty for decision making:
  - Uncertainty
  - Complexity of Environment or of System
  - Difficulty of Measuring or even Studying alternative Strategies
- Do NOT procrastinate!
- Do NOT hide your head under the sand!
- Do NOT ignore data & evidence!

Simple pieces for advice …

- Overcome your anxieties
- Let go your inner perfectionist
- Maintain a balance between
  - Analysis/deliberation and action
  - Data gathering and obtaining of results
  - Alternative goals (that might even be conflicting) – remember: balanced scorecard
  - Quantitative vs. qualitative approaches
- Manage Meetings (Group decision making)
  - Hidden agendas, fights, late starts, topic switching, …
  - Create Shared Understanding
  - Try for Consensus at least about the problem

Can we avoid wrong decisions?

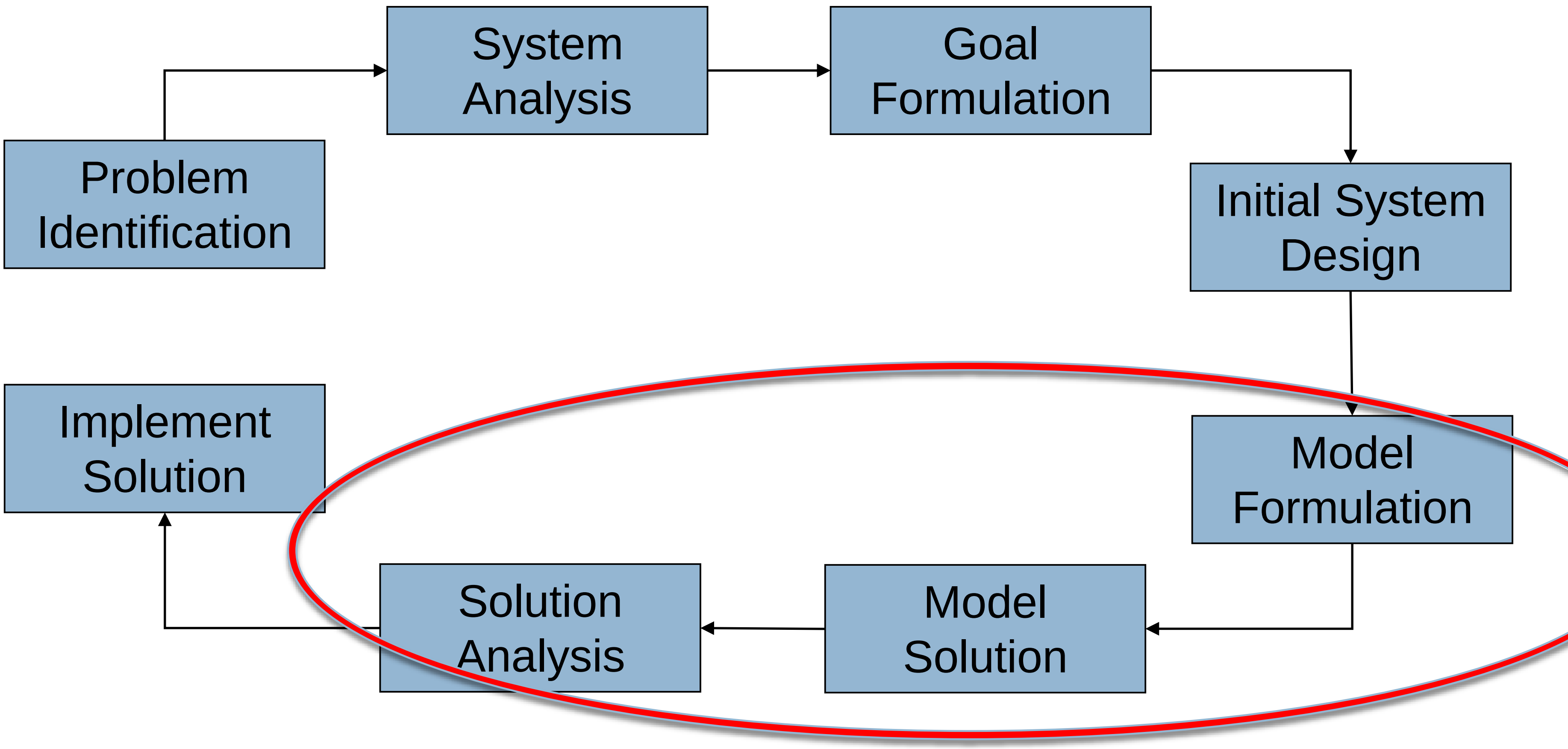
- There is NO such thing as a perfect decision maker. Even with all the supercomputers in the world, you will STILL make mistakes
- Difference between WRONG vs. BAD decision: You cannot avoid some bad decisions, but TRY to avoid the bad ones!
- Difference between outcomes and process!
- What are the key decision TRAPS?
- **Control the process**

Rule 1: Avoid the decision traps

- Status Quo trap
- Anchoring trap
- Confirming evidence trap
- And more …

- How do you avoid decision traps?

Rule 2: Follow the rational process



- The importance of analytics & modeling!

Solving the *right* problem

- Better have an approximate solution to today's problem than an optimal solution to yesterday's problem
- Make sure you get problem statement right
  - Objective (often multiple conflicting objectives)
  - Constraints (often too many)
    - Test with a known solution
- Data Quality is key
  - Garbage in, garbage out
- Make sure you always output a solution
  - Relax the problem, move constraints to objective
  - Handle the computational time
    - Trade offs among solution quality vs feasibility vs optimality

Basic concepts & decision models

Prescriptive Decision Models help decision makers identify the best solution:

Optimization - finding values of decision variables that minimize (or maximize) something such as cost (or profit).

- **Decision Variables** - the variables whose values the decision maker is allowed to choose.
- **Objective function** - the equation that minimizes (or maximizes) the quantity of interest.
- **Constraints** - limitations or restrictions that must be satisfied.

Basic concepts & decision models

- **Feasible solution** - is any set of values of the decision variables that satisfies all of the constraints.
- **Feasible region** - the set of all feasible solutions.
- **Infeasible solution** - is a solution where at least one constraint is not satisfied.
- **Optimal solution** - values of the decision variables at the minimum (or maximum) point that satisfy some necessary **optimality conditions**
- **Global and local optimality** – A local optimum is a solution that is optimal within a neighboring set of solutions. Global optimum is the optimal solution among all possible solutions

Evolution & Quality of Information

Depending on the Quality of Information:

- Deterministic models have inputs that are known with certainty.
- Stochastic models have one or more inputs that are not known with certainty.

Depending on the Evolution of Information:

- Static models where all inputs are known in advanced (with certainty or uncertainty).
- Dynamic models where input data is revealed in real time during the planning horizon.

Types of Deterministic Models

- Linear versus Nonlinear (convex optimization)
  - Linear/nonlinear functions for objective and/or constraints (LP / NLP)
- Discrete versus Continuous
  - Continuous, integer, binary and/or mixed integer decision variables (ILP / IP/ MIP / MILP / MINLP)
- Convex versus non Convex
- Quadratic Programming (QP / MIQP)
- Unconstrained: No constraints
- Dynamic Programming: Solved in stages
- Combinatorial Optimization
- …

Optimization Methods

- **Algorithms** are systematic procedures used to find optimal solutions to decision models.
- **Exact Mathematical Programming algorithms** provide guarantee for finding the (global) optimum
  - Simplex, Interior Point (Barrier), Complete Enumeration, Branch and Bound/Cut/Price, Gradient Methods…
- **Heuristic algorithms** trade optimality for efficiency and they are used to find high quality solutions in a reasonable amount of time.
  - Construction heuristics, Local Search, Evolutionary and Genetic Algorithms, Swarm Intelligence…

A simple problem for you …

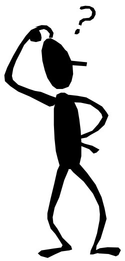
- You start working tomorrow as a Production Manager in a Manufacturing Company:

- ✓ Producing 50 different products (or variations)
- ✓ Out of 15 resources (raw materials, machine -hours, man -hours by speciality, ...)
- ✓ All 50 products PROFITABLE !
- ✓ Whatever quantity we make of each product (within our capacities) can be sold (we are small compared to the size of the market)

- ➢ Our ONLY criterion (to start with) is PROFITABILITY (i.e. no market share, ...)

Question:

- How many out of the 50 products would you consider reasonable to produce ?



All 50? 40? 30? 15? 5? 1?

Do you need more info to answer?

WHY?

Why? Consider a simple case …

- ✓ A company is producing 50 products out of 1 resource
- ✓ Product P_i requires Q_i units of the resource to be produced
- ✓ 1 unit of product P_i generates revenue R_i

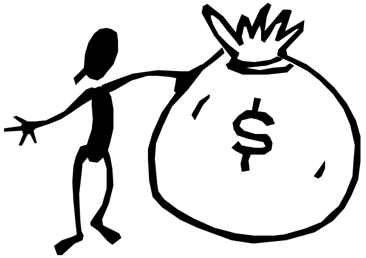
	Products			
	1	2	…..	50
Q	Q_1	Q_2		Q_{50}
R	R_1	R_2		R_{50}

How many products to produce?

- Assume that all production can be sold!
- Our objective is to max revenue!

☞ Best strategy is to produce the ONE product that maximizes the ratio:

$$R_i \,/\, Q_i$$



(REMEMBER: BEST VALUE FOR MONEY)

Extension …

- To two resources
- Generally

KEY CONCLUSIONS

☞ Do NOT spread yourself too thin!

☞ Put your resources where they will generate the most output!

☞ This is a result of the Linearity assumption

☞ Use as a YARDSTICK!

Generally, Optimization …

- … helps businesses make **complex decisions** and **trade-offs about limited resources**
  - Discover previously unknown options or approaches
  - Automate and streamline decisions
    - Compliance with business policies and regulations
  - Explore more scenarios and alternatives
    - Understand trade-offs and sensitivities to various changes
    - Gain insights into input data
    - View results in new ways

Introduction to Linear Programming Optimization

An Example

- Manufacturing company
- Producing 2 products: P1 & P2
- Out of 3 raw materials: A, B, C
- Products sell for 200 & 300 euros per unit
- Company has available stock for 30, 20 and 36 units
- Bill-of-materials:

Raw Material	Products		Available Stock
	P1	P2	
A	1	2	30
B	1	1	20
C	2	1	36
Price	200	300	

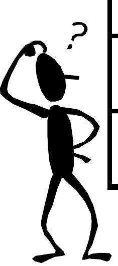
Question: What is the best production plan?

(i.e. maximizing revenue)

Try alternative solutions?

- $P_1 = 20$ $P_2 = 20$?
- $P_1 = 20$ $P_2 = 10$?
- $P_1 = 10$ $P_2 = 20$?
- $P_1 = 15$ $P_2 = 5$?
- $P_1 = 5$ $P_2 = 15$?
- $P_1 = 5$ $P_2 = 12$?

Raw Material	Products		Available Stock
	P1	P2	
A	1	2	30
B	1	1	20
C	2	1	36
Price	200	300	



<u>Questions:</u> **Are the above plans feasible?**

If feasible, are they the best?

Formulation of a model (3 steps) …

A. Determine decision variables

x_1 = production quantity of P1

x_2 = production quantity of P2

B. Determine objective : MAX REVENUE (Z)

$$Z = 200x_1 + 300x_2$$

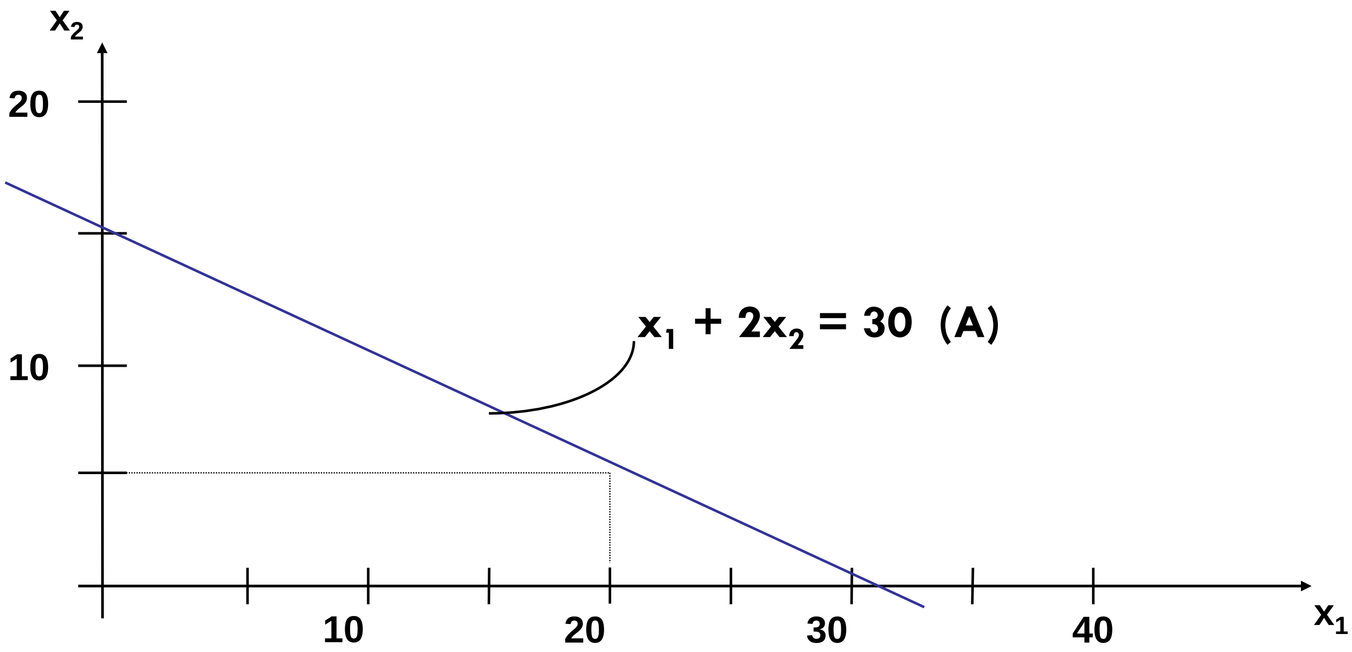
C. Determine constraints (Limited Resources)

Limited A $\rightarrow$ $x_1 + 2x_2 \leq 30$

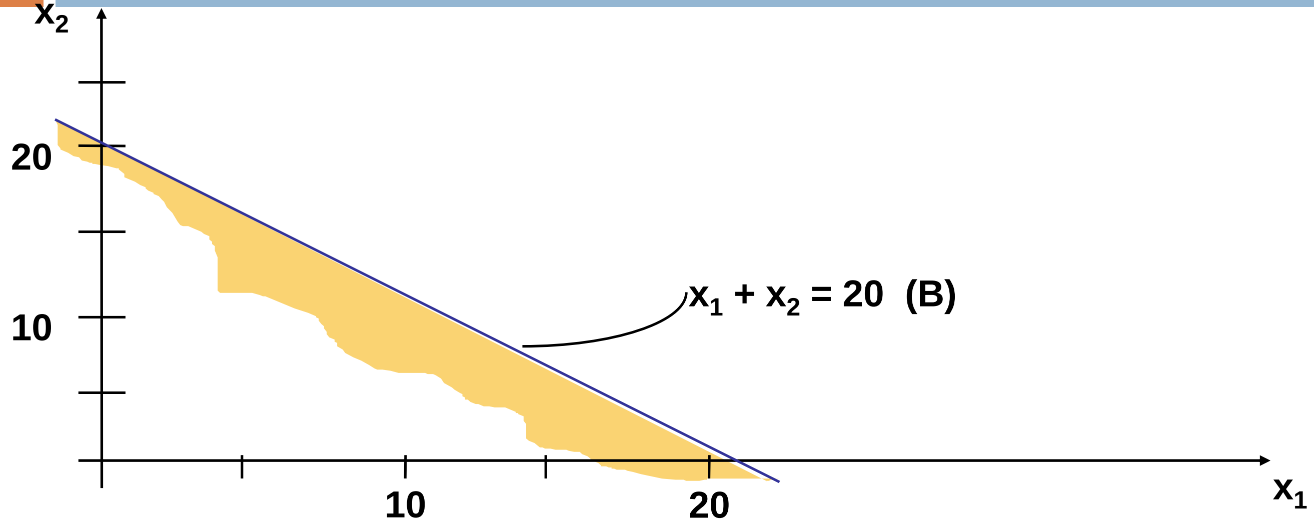
Limited B $\rightarrow$ $x_1 + x_2 \leq 20$

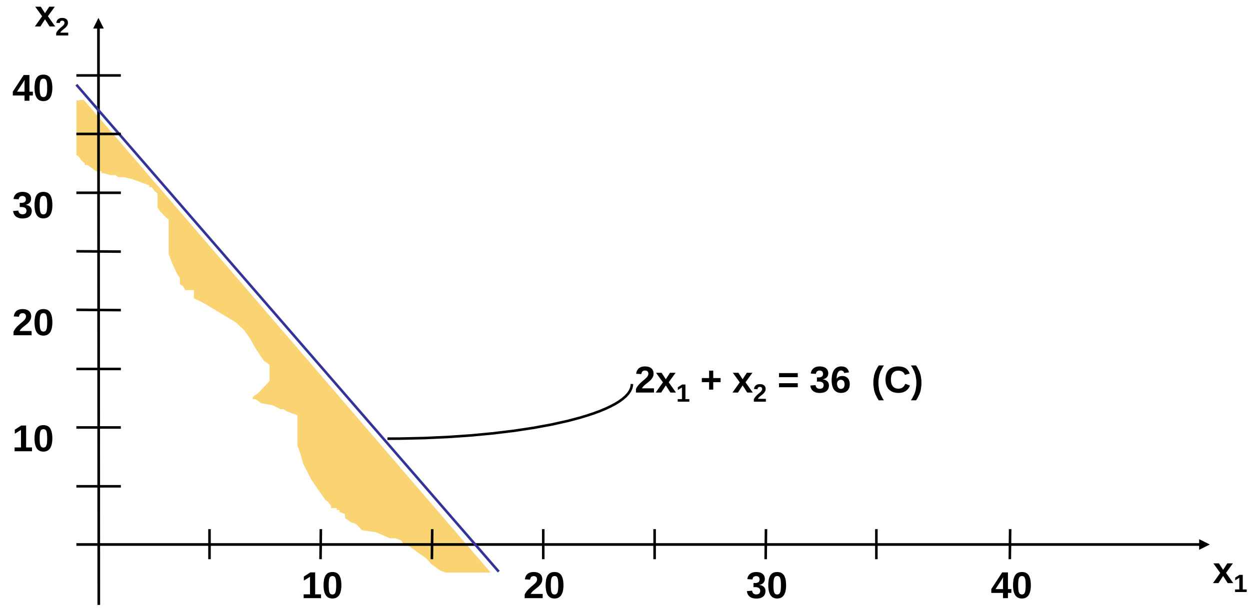
Limited C $\rightarrow$ $2x_1 + x_2 \leq 36$ $\quad x_1, x_2 \geq 0$

Graphical representation of a constraint

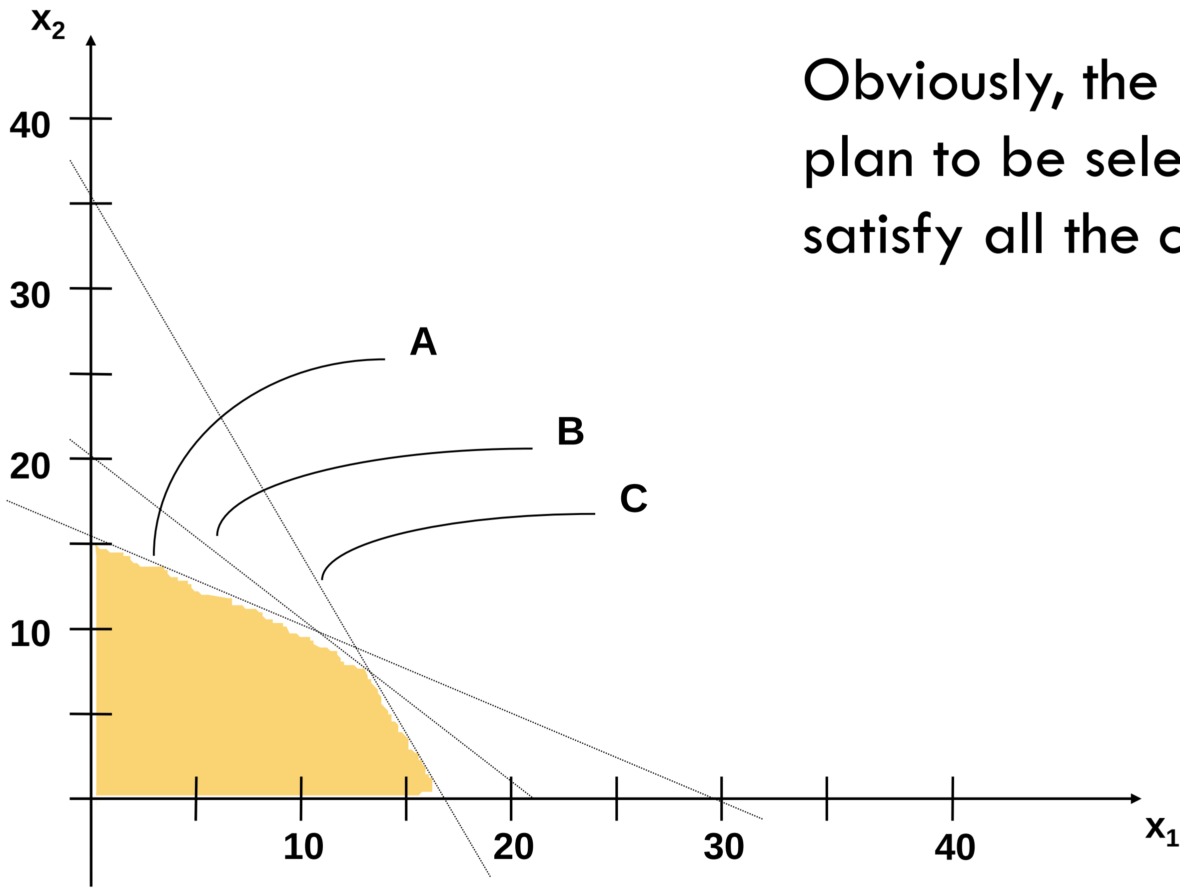


Graphical Analysis of Constraints



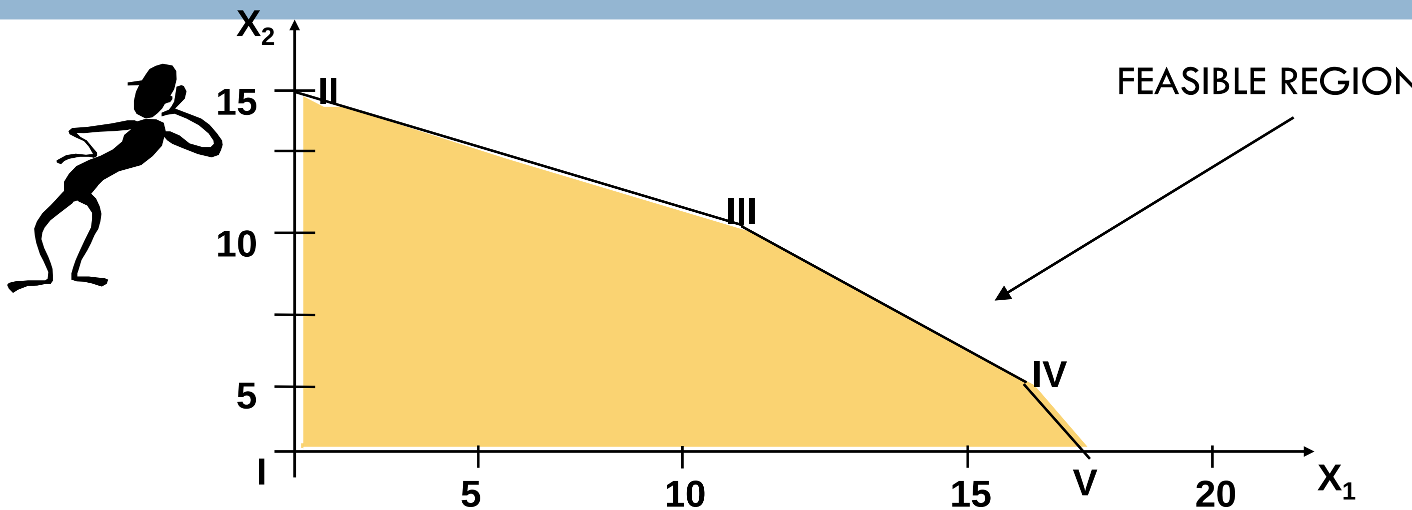


Putting all constraints together



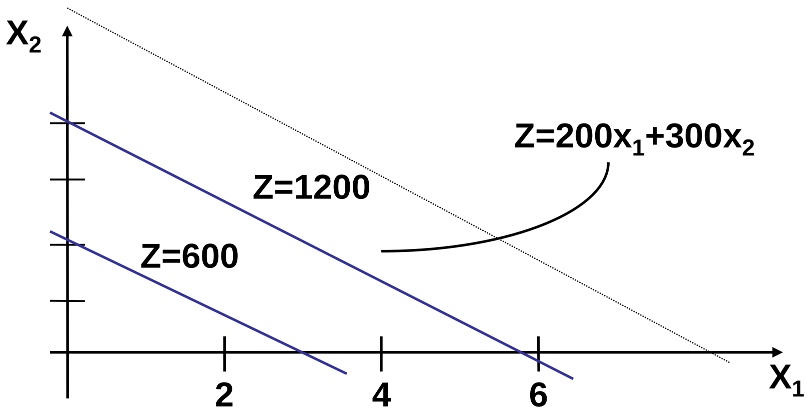
Obviously, the production plan to be selected must satisfy all the constraints.

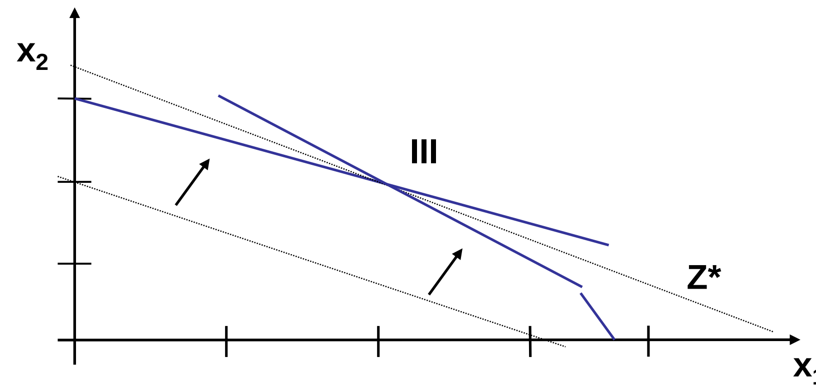
The FEASIBLE region



1. All points **within** the shaded area satisfy the constraints with **Inequality**, i.e. leave **slack** resources!
2. All points **on** the boundaries (except the corner points) utilize **one** resource completely, but leave **slack** resources of the other two!
3. The **corner points** III and IV utilize **two** resources fully!

Getting to the optimal plan





- Therefore, the OPTIMAL PRODUCTION is given by point III which is the intersection of constraints (A) & (B)!
- At this CORNER POINT, resources (A) & (B) are FULLY UTILIZED, whereas resource (C) is not!

To determine point III, solve A & B <u>as equalities</u>, simultaneously:

$$\left.\begin{aligned} x_1 + 2x_2 &= 30 \\ x_1 + x_2 &= 20 \end{aligned}\right\} \Longrightarrow \quad x_1^* = x_2^* = 10\ ;\ Z^* = 5{,}000$$

Sensitivity Analysis

What happens if the prices of two products change?

➢Assume the objective function is:

$$Z = c_1x_1 + c_2x_2 \qquad (\text{with } c_1 = 200,\ c_2 = 300)$$

➢By changing the prices c_1 and c_2, **the <u>slope</u>, of the objective function <u>changes</u>:**

- For *small variations*, same optimum remains!
- For *large variations*, the optimum "moves" to a "neighboring" corner
- The critical factor is not the values of the prices, but their relative ratio

Sensitivity Analysis

Specifically:

If……………………$c_1/c_2 \leq ½$………………optimal is II
If………………$½ \leq c_1/c_2 \leq 1$………………optimal is III
If………………$1 \leq c_1/c_2 \leq 2$………………optimal is IV
If………………$2 \leq c_1/c_2$………………………optimal is V

NOTE:

1. Optimal point is ALWAYS A CORNER POINT !
2. Even if price of P1 (c_1) increases by 30%, WE DO NOT produce more of P1 !!!
3. When the price of P1 exceeds that of P2 (i.e. $c_1/c_2 \geq 1$), only then do we change the production plan, and we change it drastically.

The new production plan will be at corner IV [utilizing (B) and (C) with $x_1 = 16$ and $x_2 = 4$!!!

Sensitivity Analysis

What happens if the <u>availabilities</u> of the resources change?

Assume that availability of A becomes 29 instead of 30

•How do you expect the strategy to change?

•How do you expect the bottom line to change?

By changing the prices b_A the feasible region "decreases"!

Even for small variations of the critical resources, the strategy changes!

So does the bottom line!

This is radically different from the previous result (prices)!

General Formulation of LP Problems

Determine the values $(x_1, x_2, ..., x_n)$ for activities $(1, 2, ..., n)$

so as to: $\text{Max } Z = c_1x_1 + c_2x_2 + ... + c_nx_n$

subject to:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &\leq b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &\leq b_m \\ x_1, x_2, ..., x_n &\geq 0 \end{aligned}$$

<u>Variations of the Above Standard Form</u>

- Minimization objective
- Constraints of the form $\geq$, or Constraints of the form $=$
- Variables $x_i \leq 0$, or Variables x_i without constraint

Assumptions of Linear Programming

1. LINEARITY
2. DIVISIBILITY
3. CERTAINTY

Linearity

- ☞ Linear graph vs. non-linear graph
- ☞ Proportionality of output to the input
- ☞ Examples of linear relationships
  - ✓ production of products vs. time
  - ✓ transportation costs vs. weight (usually)
  - ✓ distance traveled vs. time (with const. speed)
- ☞ Examples of non-linear relationships
  - ✓ Economics of scale
  - ✓ The “typist example”
  - ✓ transportation costs with economies or discounts
- ☞ Examples of piecewise linear relationships
  - ✓ approximations to non-linear relationships

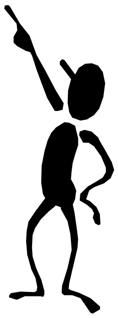
Linear Relation Input - Output

Always remember: The change in the output for a given change of input is constant at every value of the input

Example:

$y = 20 - 4x_1 + 2x_2$

x_1	y	Δx_1	Δy
0	$20+2x_2$	-	-
1	$16+2x_2$	1	-4
2	$12+2x_2$	1	-4
3	$8+2x_2$	1	-4
4	$4+2x_2$	1	-4
5	$2x_2$	1	-4



Divisibility

Activities can take any value, i.e. not necessarily integers

Examples:
- ⇨ hours of operation of a machine
- ⇨ gallons of petrol
- ⇨ pounds of wheat
- ⇨ budget, etc.

Integer variables:
- ⇨ no. of branches of a bank
- ⇨ no. of employees
- ⇨ no. of books printed, etc.

In this case use INTEGER PROGRAMMING.

Note: Even then, approximation using LP is possible if the values are big!
In this case ... round-off to the nearest integer.
Approximation is not possible for 0/1 problems !!!

Certainty

All parameters of the problem are known, i.e.

⇨ Constraints in the objective function (prices)
⇨ RHS (availabilities)
⇨ Constants of the matrix (specifications of production)

Note: Even if not known, we can do a Sensitivity Analysis !

Optimization Overview

- Variables: $$x = (x_1, x_2, \ldots, x_N)$$
- Objective: $$\min \; f(x)$$
- Subject to Functional and Regional Equations and Constraints:
  - Sometimes additional constraints:
    - Binary
    - Integer
  - Sometimes *uncertainty* in parameters (stochastic optimization)

$$\begin{cases} x \in X \\ g(x) = b \\ h(x) \geq k \\ \ldots \end{cases}$$

Example: Asset & Liability Management

- Loans
  - ◊ short-term (12%)
  - ◊ medium-term (10%)
  - ◊ long-term (8%)
- Stocks
  - ◊ average return (15%)
- Oblig. deposits at the Central Bank
  - ◊ interest = 4%
- Cash

- Current Accounts
  - ◊ total available = $10bn
- Deposit Accounts
  - ◊ total available = $45 bn
- Deposits: time deposits
  - ◊ total available =$45 bn

Special considerations

a) Liquidity:

5%, 3% and 1% of available for each category of deposits

b) Loans Limitations:

short-term between 10% and 15% of total deposits

long-term between 15% and 20% of total deposits

c) Obligatory Deposits (to the Central Bank):

at least 8% of total deposits

Determine the optimal structure of the Bank's assets
(to max total Return on Assets)

The model

Let x_i = % of total deposits placed in asset i

i = 1	short term loans
i = 2	medium-term loans
i = 3	long-term loans
i = 4	stocks
i= 5	obligatory placements
i = 6	cash

$$\text{Max } Z = 0.12x_1 + 0.10x_2 + 0.08x_3 + 0.15x_4 + 0.04x_5$$

s.t.

$$0.10 \leq x_1 \leq 0.15$$

$$0.15 \leq x_3 \leq 0.20$$

$$x_5 \geq 0.08$$

$$x_6 \geq [(0.05)(10)+(0.03)(45)+(0.01)(45)]/100 = 0.0185\ (=1.85\%)$$

$$x_1 + x_2 + \ldots + x_6 = 1$$

$$x_i \geq 0 \qquad i = 1, 2, \ldots, 6$$

Basic economic concepts – Duality

Example: The diet problem

A Consumer's diet should include daily at least 9 vitamins A and 19 vitamins C. The Consumer visits the local supermarket and determines 6 foods (F1, ..., F6) that include these vitamins, as follows:

Vitamins	Contents per 100 gm					
	F1	F2	F3	F4	F5	F6
A	1	0	2	2	1	2
C	0	1	3	1	3	2
Price / 100 gm	35	30	60	50	25	22

Assuming that he has no preference among the 6 foods, his problem is to select the MINIMUM COST DIET!

The model

Mathematical Model

Determine $(x_1, x_2, \ldots, x_6)$ = quantities of foods to buy

$$\text{MIN } Z = 35x_1 + 30x_2 + 60x_3 + 50x_4 + 25x_5 + 22x_6$$

s.t.

$$\begin{aligned} x_1 + 2x_3 + 2x_4 + x_5 + 2x_6 &\geq 9 \\ x_2 + 3x_3 + x_4 + 3x_5 + 2x_6 &\geq 19 \\ x_1, \ldots, x_6 &\geq 0 \end{aligned}$$

Solution

$x_5 = 5, x_6 = 2, x_1 = \ldots = x_4 = 0$ $\qquad Z = 169$

Some questions…

1. Why not buy any of the other foods?
2. What would it take to buy them?
3. What happens if the doctor changes the prescription?
4. What is competition, and what would competition do?

Vitamins	Contents per 100 gm					
	F1	F2	F3	F4	F5	F6
A	1	0	2	2	1	2
C	0	1	3	1	3	2
Price / 100 gm	35	30	60	50	25	22

The competition

- The pharmacist
- What is his/her objective?
- What are his/her constraints?
- Determine prices P_A and P_C for the vitamins, so that he/she maximizes his/her revenue while staying competitive!
- Can we formulate an LP to solve it?

The model

Determine (P_A, P_C) = prices of vitamins A, C

$$\text{MAX } \Theta = 9P_A + 19P_C$$

$$
\begin{array}{llcll}
\text{s.t.} & P_A & \leq & 35 & (F_1) \\
 & P_C & \leq & 30 & (F_2) \\
 & 2P_A + 3P_C & \leq & 60 & (F_3) \\
 & 2P_A + P_C & \leq & 50 & (F_4) \\
 & P_A + 3P_C & \leq & 25 & (F_5) \\
 & 2P_A + 2P_C & \leq & 22 & (F_6) \\
 & P_A \;, P_C & \geq & 0 &
\end{array}
$$

The solution

- $P_A = 4$ . . Dual Price of vitamin A
- $P_C = 7$ . . Dual Price of vitamin C
- $\Theta = 169$ . . Max Revenue
- Can you explain this?
- What is the meaning of these dual prices?
- What do they mean to the customer's budget?
- What are the surplus costs?

Dual Prices and Surplus Costs

- Dual prices give us the change in the objective function value if the RHS changes by 1 unit!
  - For how long is this dual price valid?
  - Is the dual price increasing or decreasing as we increase the RHS (availability)? Why?
- Surplus cost gives us the change that has to occur to a non-basic variable to become basic!
  - Remember: if $x>0$ … then … Surplus cost = 0, AND if $x = 0$ … then … Surplus cost > 0 … **Can you explain?**
  - Is it clear how many basic variables we will have?
- Sensitivity analysis gives us the range of values in the RHS (or in the objective function) where the strategy (or the dual prices remain constant.

The charcoal example

Three important managerial questions

- Suppose next week the doctor changes the prescription from 9 A's and 19 C's to 10 A's and 19 C's. Will this change the budget of the consumer? If so, by how much?
- By how much should the supermarket lower the prices of the vitamins not sold, in order to make them attractive?
- Suppose there is a new food with 3 A's, and 5 C's which should sell at 59c. Should the supermarket get this new food?

Dual Prices

- The dual price of a resource is:
  - Internal Value of this resource!
  - How much it is worth to us!
  - How much the objective function would increase if we had 1 more unit available!
- The dual price depends:
  - On the availability of this resource
  - On the efficiency of our technology
  - On the brand name and the prices of our finished products
- Dual prices of the same resource are different across users, and the dual price is different from its price

The Dual Price = The Real Value

- It is very important to know the dual price of a resource:
  - We know how much and at what price we should buy them in the market
  - We determine our "really valuable" resources
  - We identify good opportunities to buy
  - We can "price-out" new products or activities and calculate their profitability
  - Of course, we can also "price-out" existing products or activities
  - Example: a new food appears with 4A's and 2C's, costing 47c – would the customer prefer it?
  - REMEMBER: A DUAL PRICE FOR EVERY CONSTRAINT!

Surplus Cost

- The surplus cost of a <u>variable</u> is the change required in the price of this product …
  - … to make it competitive in the market!
  - … to start buying it!
  - … to raise its activity level to zero!
  - … to make this variable BASIC!
- A Surplus cost exists ONLY if the variable is zero!
  - Otherwise, its activity level is already positive!

Surplus Cost = Extra Cost

- It is “connected” to the dual price…
  - We can validate the surplus cost of an activity or a product by calculating the difference

Surplus Cost =

= Price of a product – “Priced-out” sum of values of its resources

- It is very important we know the surplus costs of our products… this way we will know:
  - How much we can reduce prices!
  - Which are the “hopeless” products!
  - How we compare with competition! … and, what happens if competition’s prices are lower, even after the reduction of surplus cost?

Remember

- Every constraint is associated with a DUAL PRICE
  - Try to understand what it means for every constraint …
  - Can the dual price be zero? When?
- Every variable is associated with a SURPLUS COST
  - Try to understand what it means for every variable …
  - Can the surplus cost be zero? When?
- Dual prices and surplus costs ARE RELATED
  - … though pricing out …
- Dual prices and Surplus costs can be read out of the SOLVER output

Sensitivity Analysis for LPs

- An excellent way to address UNCERTAINTY using LP
- Often it is useful to perform sensitivity analysis to see how (or if) the optimal solution changes as one or more inputs change.
- The Solve dialog box offers you the option to obtain a sensitivity report.
- Solver's sensitivity report performs two types of sensitivity analysis:
  1. on the coefficients of the objectives, the *c*'s, and
  2. on the right hand sides of the constraints, the *b*'s.

Example: A Transportation problem

- Oil is to be transported from 4 refineries (A, B, C, D) to 3 depots (1, 2, 3)
- The availability in each refinery (in tanker loads) is: 22, 41, 27, 10
- The demand to be satisfied at every depot (in tanker loads) is: 30, 45, 14

Depots

Refinery	1	2	3
A	90	30	120
B	60	60	90
C	30	180	60
D	120	150	30

From the SOLVER output

Target Cell (Min)

Cell	Name	Original Value	Final Value
$E$9	Cost	0.00	3780.00

Adjustable Cells

Cell	Name	Original Value	Final Value
$B$5	Refinery 1 Depot 1	0.00	0.00
$C$5	Refinery 1 Depot 2	0.00	22.00
$D$5	Refinery 1 Depot 3	0.00	0.00
$B$6	Refinery 2 Depot 1	0.00	3.00
$C$6	Refinery 2 Depot 2	0.00	23.00
$D$6	Refinery 2 Depot 3	0.00	5.00
$B$7	Refinery 3 Depot 1	0.00	27.00
$C$7	Refinery 3 Depot 2	0.00	0.00
$D$7	Refinery 3 Depot 3	0.00	0.00
$B$8	Refinery 4 Depot 1	0.00	0.00
$C$8	Refinery 4 Depot 2	0.00	0.00
$D$8	Refinery 4 Depot 3	0.00	10.00

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$F$12	Availability of refinery 1 Used	22.00	$F$12<=$G$12	Binding	0
$F$13	Availability of refinery 2 Used	31.00	$F$13<=$G$13	Not Binding	10
$F$14	Availability of refinery 3 Used	27.00	$F$14<=$G$14	Binding	0
$F$15	Availability of refinery 4 Used	10.00	$F$15<=$G$15	Binding	0
$F$16	Requirements of depot 1 Used	30.00	$F$16=$G$16	Not Binding	0
$F$17	Requirements of depot 2 Used	45.00	$F$17=$G$17	Not Binding	0
$F$18	Requirements of depot 3 Used	15.00	$F$18=$G$18	Not Binding	0
$B$5	Refinery 1 Depot 1	0.00	$B$5>=0	Binding	0.00
$C$5	Refinery 1 Depot 2	22.00	$C$5>=0	Not Binding	22.00
$D$5	Refinery 1 Depot 3	0.00	$D$5>=0	Binding	0.00
$B$6	Refinery 2 Depot 1	3.00	$B$6>=0	Not Binding	3.00
$C$6	Refinery 2 Depot 2	23.00	$C$6>=0	Not Binding	23.00
$D$6	Refinery 2 Depot 3	5.00	$D$6>=0	Not Binding	5.00
$B$7	Refinery 3 Depot 1	27.00	$B$7>=0	Not Binding	27.00
$C$7	Refinery 3 Depot 2	0.00	$C$7>=0	Binding	0.00
$D$7	Refinery 3 Depot 3	0.00	$D$7>=0	Binding	0.00
$B$8	Refinery 4 Depot 1	0.00	$B$8>=0	Binding	0.00
$C$8	Refinery 4 Depot 2	0.00	$C$8>=0	Binding	0.00
$D$8	Refinery 4 Depot 3	10.00	$D$8>=0	Not Binding	10.00

Optimal Solution:

- X12 = 22
- X21 = 3
- X22 = 23
- X23 = 5
- X31 = 27
- X43 = 10
- Z = 3,780

- Why not use route 1→3?
- Would total cost change if refinery 4 had 1 more tanker load available?

From the SOLVER output

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$5	Refinery 1 Depot 1	0.00	60.00	90	1E+30	60
$C$5	Refinery 1 Depot 2	22.00	0.00	30	30	1E+30
$D$5	Refinery 1 Depot 3	0.00	60.00	120	1E+30	60
$B$6	Refinery 2 Depot 1	3.00	0.00	60	60	0
$C$6	Refinery 2 Depot 2	23.00	0.00	60	150	30
$D$6	Refinery 2 Depot 3	5.00	0.00	90	0	60
$B$7	Refinery 3 Depot 1	27.00	0.00	30	0	1E+30
$C$7	Refinery 3 Depot 2	0.00	150.00	180	1E+30	150
$D$7	Refinery 3 Depot 3	0.00	0.00	60	1E+30	0
$B$8	Refinery 4 Depot 1	0.00	120.00	120	1E+30	120
$C$8	Refinery 4 Depot 2	0.00	150.00	150	1E+30	150
$D$8	Refinery 4 Depot 3	10.00	0.00	30	60	1E+30

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$F$12	Availability of refinery 1 Used	22.00	-30.00	22	23	10
$F$13	Availability of refinery 2 Used	31.00	0.00	41	1E+30	10
$F$14	Availability of refinery 3 Used	27.00	-30.00	27	3	10
$F$15	Availability of refinery 4 Used	10.00	-60.00	10	5	10
$F$16	Requirements of depot 1 Used	30.00	60.00	30	10	3
$F$17	Requirements of depot 2 Used	45.00	60.00	45	10	23
$F$18	Requirements of depot 3 Used	15.00	90.00	15	10	5

<u>From dual prices/reduced costs</u>

- Cost of route 1→3 has to be reduced by 60!
- If R4's availability increases by 1, then the total cost goes down by 60!
- VERIFY IT!

Modelling Mixed Integer Linear Problems

Integer Programming Problems

- Allocation of workers per shift
- Production of cars per week
- Number of bank branches operating in an area

All problems of allocation of resources, when some or all of these resources or activities can be allocated or undertaken at integer values.

- Usually, these problems can be approximated using **Linear Programming,** and rounding-off to the nearest integer
- ONE exception …

Binary Problems

- A special category of Integer Problems
- The variables take **only the values 0 or 1 (binary variable)**
- These are **logical variables … not physical variables**
- They represent **logical decisions (YES/NO)**, not physical quantities
- They cannot be solved sing LP – no rounding off can occur!
- They need special "art" in the formulation
- They appear very frequently in many problems:
  - Investment evaluation and selection
  - Distribution and vehicle routing
  - Production scheduling

Example: The Knapsack Problem

- Assume you are going for a 3-day safari.
- Your knapsack can take items of total weight no more than 20 Kilos and total volume no more than 30 gallons
- You are asked to choose among 10 items (foods), each one being characterised by a weight $w_{i,}$ a volume $v_{i,}$ and a value $c_{i,}$.
- Your objective is to choose the best combination among the 10 items available, i.e. the ones that maximize the total value of your knapsack.

Knapsack problem: Solution

$$\text{Let } X_i = \begin{cases} 1, \text{ if item } i \ (i = 1\ldots10) \text{ is to be included in the knapsack} \\ 0, \text{ otherwise} \end{cases}$$

Problem Formulation:

$$\begin{array}{lllll} \text{Max} & Z = c_1X_1 + c_2X_2 + \ldots + c_{10}X_{10} & & \\ \text{s.t.} & & & \\ & w_1X_1 + w_2X_2 + \ldots + w_{10}X_{10} & \leq & 20 \\ & v_1X_1 + v_2X_2 + \ldots + v_{10}X_{10} & \leq & 30 \\ & X_1, \ldots, X_{10} = 0/1 & & \end{array}$$

Variations around Knapsack problem

1. Assume that items1 and 2 are "competitive", i.e. there is no sense in bringing both of them, since you will be consuming only one, (e.g. two types of toothpaste).

$$X_1 + X_2 \leq 1$$

2. Assume that items 3 and 4 are "complementary", i.e. if you take one with you, you also have to take the other as well, and vice versa (e.g. pasta and tomato sauce).

$$X_3 = X_4$$

3. Assume that foods 5 and 6 are "partially complementary", i.e. if you take the first (e.g. coffee), you also have to take the second (e.g. milk). This in not true the other way.

$$X_5 \leq X_6$$

<u>Note</u>: The above is an investment selection problem

Case 1: Investment Selection

You are considering how to best allocate the $4 M available to the following investments that receive some subsidy:

	INVESTMENT	COST (million $)	RETURN (million $)
1	Casino in Rhodes	2.50	4.20
2	Casino in Corfu	1.50	3.80
3	Private Airport in Corfu	0.70	0.75
4	Casino in North Evia	1.30	2.20
5	Factory in North Evia	1,40	1.90
6	Housing in North Evia	0.60	0.30

Cost = Own funds (over and above the subsidy)

Case 1: Investment Selection

- By law, the government can only acquire a licence for 1 casino.
- If the casino in Corfu is selected, then a small private airport needs to also be built (because no good transportation currently exists).
- The airport in Corfu is certainly beneficial anyway.
- A casino can not be operated in an industrial area.
- If the factory in Evia is constructed, the housing need for the workers must also be taken care of.
- But if the factory in Evia is not constructed, the houses do not need to be constructed either.
- Total budget (own funds) restricted to $4 million.

Case 1: The Model

Let $x_i = 1$ if you undertake investment i

$= 0$ otherwise

$$\text{Max } Z = 4.20x_1 + 3.80x_2 + 0.75x_3 + 2.20x_4 + 1.90x_5 + 0.30x_6$$

s.t.

$$2.50x_1 + 1.50x_2 + 0.70x_3 + 1.30x_4 + 1.40x_5 + 0.60x_6 \leq 4$$

$$x_1 + x_2 + x_4 \leq 1$$

$$x_2 \leq x_3$$

$$x_4 + x_5 \leq 1$$

$$x_5 = x_6$$

$$x_i = 0/1$$

Case 2: Distribution

- Four trucks are available to deliver milk to five grocery stores.
- The demand of each grocery store can be supplied by only truck, but a truck may deliver to more than one grocery.

Truck	**Capacity** (gallons)	**Daily Operating Cost** ($)
1	400	45
2	500	50
3	600	55
4	1100	60

Grocery	**Daily Demand (gallons)**
1	100
2	200
3	300
4	500
5	800

- Determine how to minimize the daily cost of meeting the demands of the five groceries.

Case 2: Distribution

Let $Y_i =$
- 1, if truck i (i = 1…4) operates on a particular day to deliver milk
- 0, otherwise

Let $X_{ij} =$
- 1, if truck i (i = 1..4) is used to deliver milk to grocery store j (j = 1, …, 5)
- 0, otherwise

$$\min Z = 45 \cdot Y_1 + 50 \cdot Y_2 + 55 \cdot Y_3 + 60 \cdot Y_4$$

so that

Truck 1 Capacity: $100 \cdot X_{11} + 200 \cdot X_{12} + 300 \cdot X_{13} + 500 \cdot X_{14} + 800 \cdot X_{15} \le 400 \cdot Y_1$

Truck 2 Capacity: $100 \cdot X_{21} + 200 \cdot X_{22} + 300 \cdot X_{23} + 500 \cdot X_{24} + 800 \cdot X_{25} \le 500 \cdot Y_2$

Truck 3 Capacity: $100 \cdot X_{31} + 200 \cdot X_{32} + 300 \cdot X_{33} + 500 \cdot X_{34} + 800 \cdot X_{35} \le 600 \cdot Y_3$

Truck 4 Capacity: $100 \cdot X_{41} + 200 \cdot X_{42} + 300 \cdot X_{43} + 500 \cdot X_{44} + 800 \cdot X_{45} \le 1100 \cdot Y_4$

$X_{1j} + X_{2j} + X_{3j} + X_{4j} = 1$ for every grocery store j (j = 1, …, 5)

$$X_{ij}, Y_i = 0/1$$

Example: Project Portfolio Selection

- Available Budget $100,000
- Project Specifications:

Project (i)	Budget (Ki) 1000x$	NPV (Ai) 1000x$
1	30	45
2	20	32
3	29	38
4	22	35
5	27	40
6	18	29

Project Portfolio Example

- Constraints and Limitations
  - We cannot exceed total budget
  - Projects 1 and 5 can only be selected as a pair, we cannot select separately project 1 or project 5.
  - We can select either project 3 or project 4, but we cannot select both of them
  - We cannot select more than 3 projects
  - We can select project 6, only if we have also selected project 3

Project Portfolio Example

- Objective
  - Select the projects that maximize the total NPV according to the constraints and limitations

Project Portfolio Example

Solution

- Binary Decision Variables: X_1, X_2, X_3, X_4, X_5 and X_6
  - These are variable can only take values 0 or 1. For example, if project 1 is selected then $X_1=1$; otherwise $X_1=0$.
- Model
  - $\text{Max } Z = (45X_1+32X_2+38X_3+35X_4+40X_5+29X_6)$
  - $30X_1+20X_2+29X_3+22X_4+27X_5+18X_6 <= 100$
  - $X_1 = X_5$
  - $X_3+X_4 <= 1$
  - $X_1+X_2+X_3+X_4+X_5+X_6 <= 3$
  - $X_6 <= X_3$
  - $X_i = 0$ or 1, for all $i = 1, \ldots 6$

Project Portfolio Example

Solution (using Excel Solver)

Project (i)	**Budget (Ki) 1000x$**	**NPV (Ai) 1000x$**
1	30	45
2	20	32
3	29	38
4	22	35
5	27	40
6	18	29

Binary Selection Variable Xi	**Total Budget**	**Total NPV**
1	30	45
0	0	0
1	29	38
0	0	0
1	27	40
0	0	0
3	86	**123**

Transportation Example

- A transportation model is formulated for a class of problems with the following characteristics:
  - a product is transported from a number of sources to a number of destinations at the minimum possible cost
  - each source is able to supply a fixed number of units of product
  - each destination has a fixed demand for product

Transportation Example

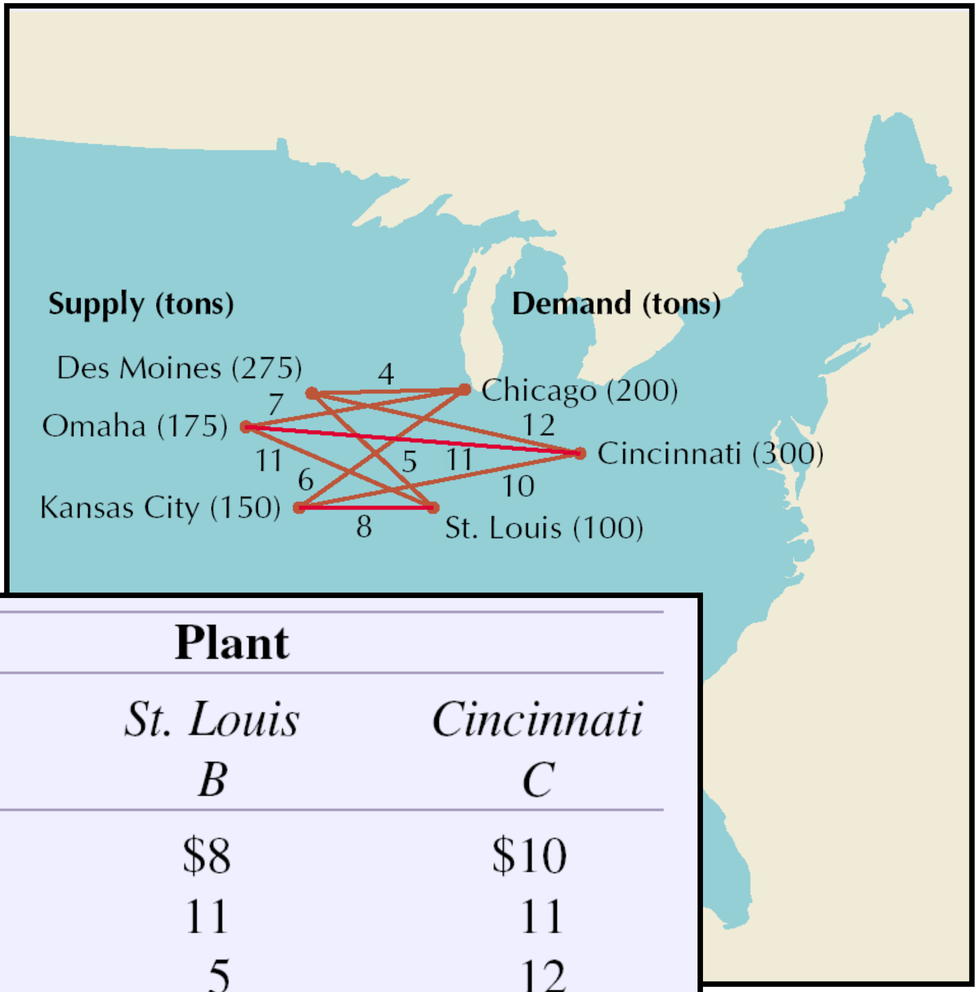
- Demand and Supply needs

Distribution Center	Supply
1. Kansas City	150
2. Omaha	175
3. Des Moines	275
	600 tons

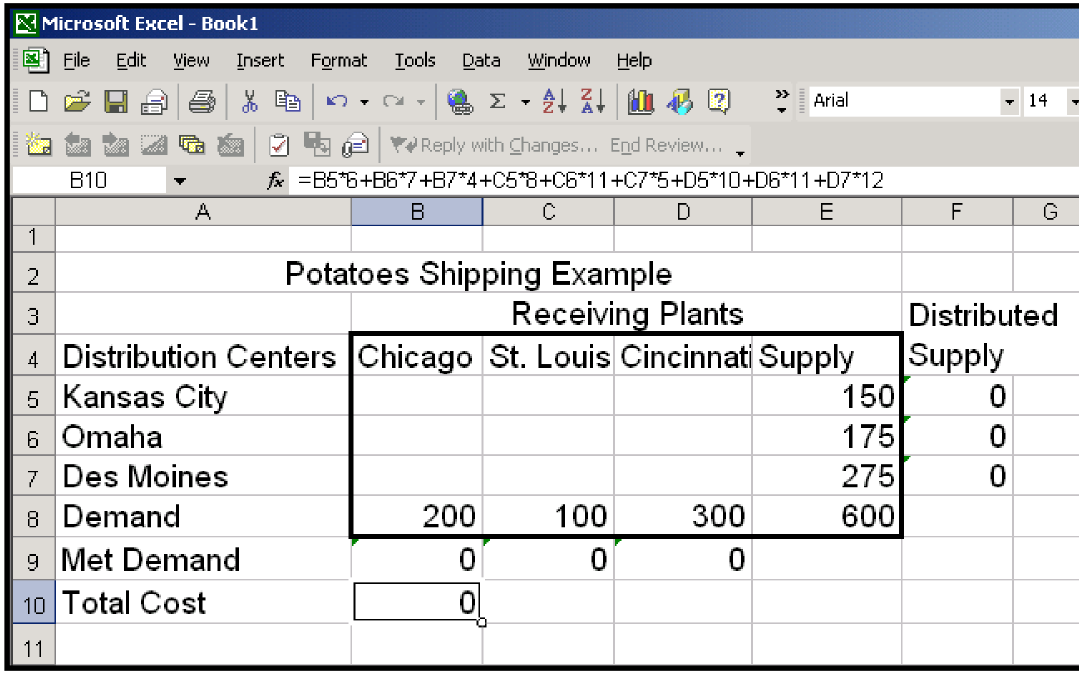
Plant	Demand
A. Chicago	200
B. St. Louis	100
C. Cincinnati	300
	600 tons

Transportation Example

- Transportation Costs among plants and distribution centers



Distribution Center	*Chicago* *A*	*St. Louis* *B*	*Cincinnati* *C*
	Plant		
Kansas City	$6	$8	$10
Omaha	7	11	11
Des Moines	4	5	12


Microsoft Excel - Book1
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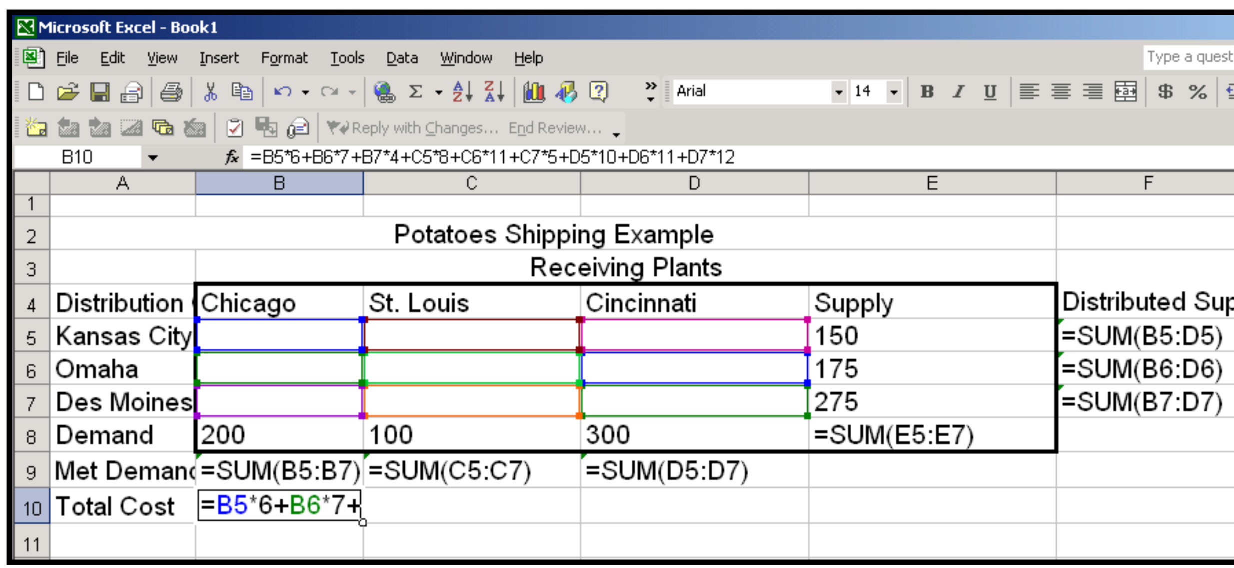
B10
=B5*6+B6*7+B7*4+C5*8+C6*11+C7*5+D5*10+D6*11+D7*12
Potatoes Shipping Example
Receiving Plants
Distributed Supply
Distribution Centers
Chicago
St. Louis
Cincinnati
Supply
Kansas City 150 0
Omaha 175 0
Des Moines 275 0
Demand 200 100 300 600
Met Demand 0 0 0
Total Cost 0


Microsoft Excel - Book1
File Edit View Insert Format Tools Data Window Help
B10
=B5*6+B6*7+B7*4+C5*8+C6*11+C7*5+D5*10+D6*11+D7*12
Potatoes Shipping Example
Receiving Plants
Distribution
Chicago
St. Louis
Cincinnati
Supply
Distributed Sup
Kansas City 150 =SUM(B5:D5)
Omaha 175 =SUM(B6:D6)
Des Moines 275 =SUM(B7:D7)
Demand 200 100 300 =SUM(E5:E7)
Met Demand =SUM(B5:B7) =SUM(C5:C7) =SUM(D5:D7)
Total Cost =B5*6+B6*7+

Transportation Example

- Solution

Potatoes Shipping Example					
	Receiving Plants				Distributed
Distribution Cer	Chicago	St. Louis	Cincinnati	Supply	Supply
Kansas City	0	0	150	150	150
Omaha	25	0	150	175	175
Des Moines	175	100	0	275	275
Demand	200	100	300	600	
Met Demand	200	100	300		
Total Cost	4525				

A simple Supply Chain

- 5 production plants (A, B, C, D, E) and 6 retail shops (1, 2, …, 6)
- Each plant has a predefined capacity, a fixed production costs and a variable production cost per unit of product produced
- Each store has a minimum supply requirement (customer demand + safety stock) and a sale price
- For the transportation of product from each plant at each retail store there is a predefined transportation cost

- Problem:
  - Find the optimum production and distribution quantities at each node
  - Maximize the profit

Simple Supply Chain

- Input Data:

	Variable Cost	Fixed Cost	Capacity	1	2	3	4	5	6
A	52	75,000	18	17	4	7	17	12	28
B	63	35,000	24	15	20	10	1	5	12
C	57	50,000	27	20	24	15	5	10	8
D	49	41,000	22	4	14	6	11	7	24
E	67	22,000	31	9	17	8	6	3	18
Demand				10	8	12	6	7	11
Price				107	125	98	115	105	130

Simple Supply Chain

- Variables:
  - $Y_A = 1$, if plant A operates, 0 otherwise
  - $Y_B, \ldots, Y_E$ 0/1 binary variables the other plants
  - A_1 = quantity that is produced in A and it is transported to retail shop 1
  - Respectively: $A_2, \ldots, A_6, B_1, \ldots, E_6 \geq 0$ continuous variables that correspond to the quantities that produced and transported to the retail shops

Simple Supply Chain

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$$\text{Max } Z = 38A_1 + 69A_2 + 39A_3 + 46A_4 + 41A_5 + \ldots + 45E_6 - -(75.000Y_A + 35.000Y_B + \ldots + 22.000Y_E)$$

s.t.

$$
\left.\begin{array}{lll}
A_1 + A_2 + \ldots + A_6 & \leq & 18\ Y_A \\
B_1 + B_2 + \ldots + B_6 & \leq & 24\ Y_B \\
\vdots & & \\
E_1 + E_2 + \ldots + E_6 & \leq & 31\ Y_E
\end{array}\right\} \text{Capacity}
$$

$$
\left.\begin{array}{lll}
A_1 + B_1 + \ldots + E_1 & \geq & 10 \\
A_2 + B_2 + \ldots + E_2 & \geq & 8 \\
\vdots & & \\
A_6 + B_6 + \ldots + E_6 & \geq & 11
\end{array}\right\} \text{Demand}
$$

$$A_1, \ldots, E_6 \geq 0$$

$$Y_A, Y_B, \ldots, Y_E = 0/1$$

Simple Supply Chain

Decision Variables						
	Retail Stores (Flow Variables)					
Plants	1	2	3	4	5	6
A	0	0	0	0	0	0
B	0	0	6	6	0	12
C	0	0	0	0	0	0
D	0	0	0	0	0	0
E	10	8	6	0	7	0

Plants	Binary variables
A	0
B	1
C	0
D	0
E	1

	Objective Function Coefficients						
	1001	1194	921	1081	986	1220	
	992	1167	907	1086	982	1225	
	993	1169	908	1088	983	1235	
	1017	1187	925	1090	994	1227	
	994	1166	905	1077	980	1215	
	0	0	0	0	0	0	
	0	0	5442	6516	0	14700	
	0	0	0	0	0	0	
	0	0	0	0	0	0	
	9940	9328	5430	0	6860	0	
Totals	9940	9328	10872	6516	6860	14700	1216
							Target Cell

Capacity Constraints		
0	<=	0
24	<=	24
0	<=	0
0	<=	0
31	<=	31
Demand Constraints		
10	>=	10
8	>=	8
12	>=	12
6	>=	6
7	>=	7
12	>=	11

Using Excel Solver

Useful tips

Using Excel Solver

- Two steps:
  - **Model development** – decide what the decision variables are, what the objective is, which constraints are required and how everything fits together
  - **Optimize** – systematically choose the values of the decision variables that make the objective as large or small as possible and cause all of the constraints to be satisfied.

Using Excel Solver

- Excel terminology for optimization
  - Decision variables = **changing cells**
  - Objective = **target cell**
  - **Constraints** impose restrictions on the values in the changing cells.
- A common form for a constraint is **nonnegativity**
- **Nonnegativity** constraints imply that changing cells must contain nonnegative values.

Using Excel Solver

- Real-life problems are almost never exactly linear. However, a linear approximation often yields very useful results.
- In terms of Solver, if the model is linear the Assume Linear Model box must be checked in the Solver Options dialog box.
- Check the Assume Linear Model box even if the divisibility property is violated.

Using Excel Solver

- If the Solver returns a message that “the condition for Assume Linear Model are not satisfied” it
  - can indicate a logical error in your formulation.
  - can also indicate that Solver erroneously thinks the linearity conditions are not satisfied.
- Try not checking the Assume Linear model box and see if that works. In any case it always helps to have a well-scaled model.

Using Excel Solver

Infeasibility and Unboundedness

- It is possible that there are no feasible solutions to a model. There are generally two possible reasons for this:
    1. There is a mistake in the model (an input entered incorrectly) or
    2. the problem has been so constrained that there are no solutions left.
- In general, there is no foolproof way to find the problem when a "no feasible solution" message appears.

Using Excel Solver

- A second type of problem is **unboundedness.**
- Unboundedness is that the model can be made as large as possible. If this occurs it is likely that a wrong input has been entered or forgotten some constraints.
- Infeasibility and unboundedness are quite different. It is possible for a model to have no feasible solution but no realistic model can have an unbounded solution.